

# A New Multiplier Inequality and Applications



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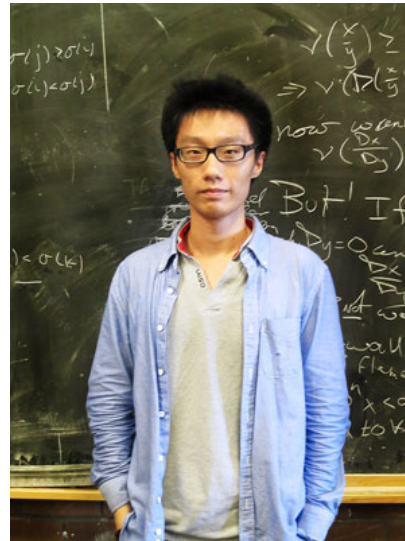
*Meeting in Honor of Aad van der Vaart's 60th Birthday*

*17-21 June 2019*

## New Multiplier Inequalities

Jon A. Wellner, Meeting in Honor of Aad van der Vaart  
Leiden, **17 June 2019**

Based on the U.W. dissertation work of [Qiyang \(Roy\) Han](#)



# Outline

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- 0. Introduction: Aad van der Vaart and two books
- 1. The current motivation:  
(nonparametric) least squares regression
- 2. The Classical Multiplier inequality and Multiplier CLT's
- 3. A new multiplier inequality
- 4. Consequences for Least Squares Regression
- 5. Further consequences (and further problems)
- 6. References

## 0. Introduction: Aad and two books

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Aad and Marie Huskova, Maastricht, 1985



Aad, Maryse, and Pascal, Leiden, 2004

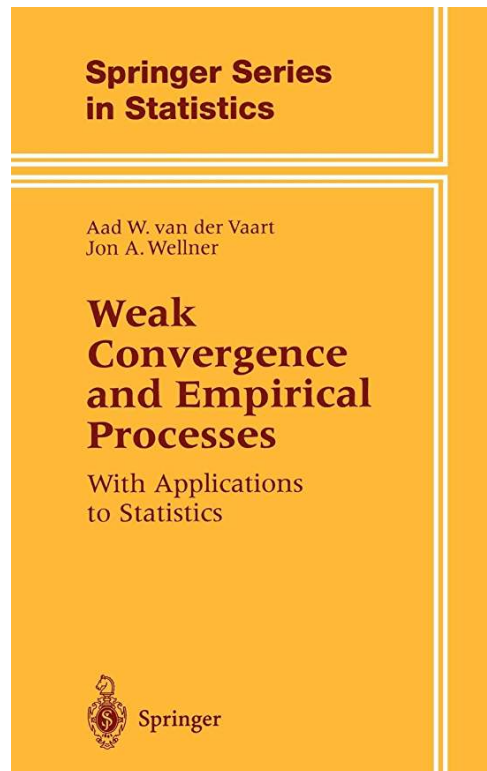


Aad; Seattle, 2010

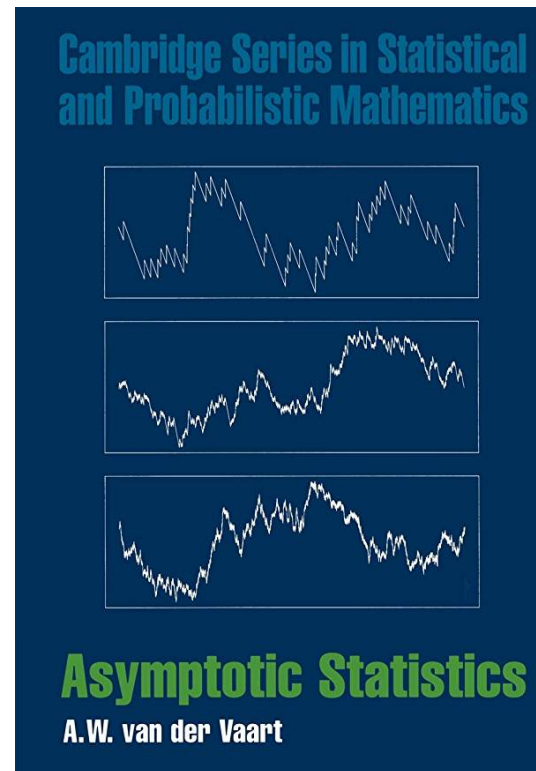
# 1. Two books:

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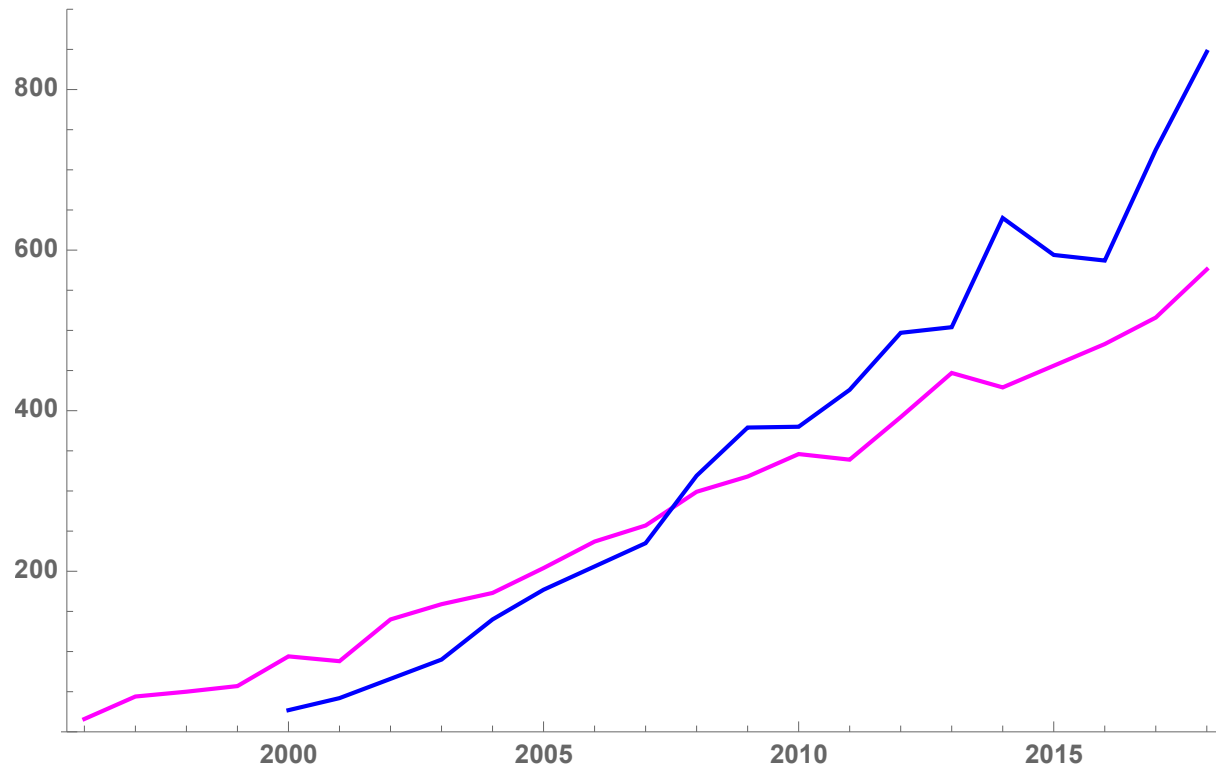
- van der Vaart & W (1996):  
*Weak Convergence and Empirical Processes.*
- van der Vaart (1998): *Asymptotic Statistics.*



WCES



AS



Number of citations by year, WCES magenta, AS blue,  
1996 - 2018

# 1. Motivation: nonparametric Least Squares

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Suppose we observe:  $(X_i, Y_i)$ ,  $i = 1, \dots, n$ ,  
where we assume that

$$Y_i = f_0(X_i) + \xi_i, \quad i = 1, \dots, n$$

and we estimate  $f_0$  by

$$\hat{f}_n \equiv \operatorname{argmin}_{f \in \mathcal{F}} \sum_{i=1}^n (Y_i - f(X_i))^2$$

where  $\mathcal{F}$  is some class of regression functions chosen by the statistician.

**Q1** Does  $\|\hat{f}_n - f_0\|_{L_2(P_X)} = O_p(\delta_n)$  for some  $\delta_n \searrow 0$ ?

**Q2** How is  $\delta_n$  determined by  $\mathcal{F}$  and the distribution of the “errors”  $\xi_i$ ?

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**Standard approach:** Basic inequality. Since  $\hat{f}_n$  is a minimizer,

$$\|Y - \hat{f}_n\|_n^2 \leq \|Y - f_0\|_n^2. \quad (1)$$

But

$$\begin{aligned} \|Y - \hat{f}_n\|_n^2 &= \|Y - f_0 + f_0 - \hat{f}_n\|_n^2 \\ &= \|Y - f_0\|_n^2 + \frac{2}{n} \sum_{i=1}^n \xi_i(f_0 - \hat{f}_n)(X_i) \\ &\quad + \|f_0 - \hat{f}_n\|_n^2 \end{aligned}$$

Using this identity in (1) yields, after cancelling  $\|Y - f_0\|_n^2$  on both sides and re-arranging,

$$\begin{aligned} \|\hat{f}_n - f_0\|_n^2 &\leq \frac{2}{n} \sum_{i=1}^n \xi_i(\hat{f}_n - f_0)(X_i) \leq 2 \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \xi_i(f - f_0)(X_i) \\ &\equiv 2 \sup_{f \in \mathcal{F}} \textcolor{blue}{MEP}_n(f) \\ &= 2 \cdot \text{supremum of a } \textcolor{blue}{\text{multiplier empirical process}}. \end{aligned}$$

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Let  $\mathbb{Z}_n(f) \equiv \sum_{i=1}^n Z_i(f)$  ,  $f \in \mathcal{F}$ , be the empirical process.  
Then for some random variables  $\xi_1, \dots, \xi_n$  independent of  $\{Z_i(f)\}$ ,  
the multiplier empirical process is:

$$\mathbb{Z}_n^\xi(f) \equiv \sum_{i=1}^n \xi_i Z_i(f), \quad f \in \mathcal{F}.$$

### Other multiplier empirical processes:

- Rademacher multipliers:  $\xi_i$  i.i.d.  $\text{Uniform}\{-1, 1\}$ .
- Gaussian multipliers:  $\xi_i$  i.i.d.  $N(0, 1)$ .
- Regression errors:  $\xi_i$  i.i.d.  $E\xi_i = 0$ ,  $\text{Var}(\xi_i) < \infty$ .
- Banach space CLT:  $\xi_i$  i.i.d.  $E\xi_i = 0$ ,  $\|\xi_i\|_{2,1} < \infty$
- Efron bootstrap:  $(\xi_1, \dots, \xi_n) \sim \text{Mult}_n(n, (1/n, \dots, 1/n))$ .
- Horvitz-Thompson finite sampling:  
inverse probability sampling weights.

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**Rademacher multipliers:** symmetrization and de-symmetrization inequalities

$$\frac{1}{2}E\left\|\sum_{i=1}^n \epsilon_i Z_i\right\|_{\mathcal{F}} \leq E\left\|\sum_{i=1}^n Z_i\right\|_{\mathcal{F}} \leq 2E\left\|\sum_{i=1}^n \epsilon_i Z_i\right\|_{\mathcal{F}}.$$

**Gaussian multipliers:** conditionally on the  $Z_i$ 's, the multiplier empirical process is a **Gaussian** process

- **Conditional multiplier CLT's:** Ledoux and Talagrand (1988).
- **Bootstrap CLT's and Multiplier Bootstraps**  
Giné and Zinn (1990), Praestgaard & W (1993).

## 2. The Classical Multiplier inequality” and Multiplier CLT’s

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- Let  $Z_1, \dots, Z_n$  be i.i.d. with values in  $B$  (a Banach space);
- Let  $\xi_1, \dots, \xi_n$  be i.i.d. real valued with  $E\xi_1 = 0$ ,  $E\xi_1^2 = 1 < \infty$ .
- Let

$$S_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i = \text{empirical process}$$

$$S_n^\xi \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i Z_i = \text{multiplier empirical process}$$

- **Q3** Suppose that  $S_n \rightarrow_d \mathbb{S} \sim \text{Gaussian in } B$ .  
(i.e.  $Z \in CLT(B)$ ). When do we have

$$S_n^\xi \rightarrow_d \mathbb{S} \text{ in } B? \tag{2}$$

(i) Unconditionally? (ii) Conditionally given the  $\{Z_i\}'s$ ?

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**A3** Giné and Zinn (1984),

Ledoux and Talagrand (1986, 1991).

(2) holds if  $E\xi_1 = 0$ ,  $E\xi_1^2 = 1 < \infty$ , and  $\xi_1 \in L_{2,1}$ :

$$\|\xi_1\|_{2,1} \equiv \int_0^\infty \sqrt{P(|\xi_1| > t)} dt < \infty. \quad (3)$$

Note:  $L_{2+\epsilon} \subset L_{2,1} \subset L_2$ .

**Q4** Is the  $L_{2,1}$  condition “sharp”?

**A4** Ledoux and Talagrand (1986). Yes!

If  $\xi_1 \notin L_{2,1}$ , then there exists a Banach space  $B$  and a probability distribution on  $B$  so that  $S_n$  satisfies the CLT in  $B$ , but the multiplier CLT (2) **fails**:  $\|S_n^\xi\| \rightarrow_p \infty$ .

For  $p \geq 1$ ,

$$\|\xi\|_{p,1} \equiv \int_0^\infty (P(|\xi| > t))^{1/p} dt.$$

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## Notation for statistics:

- $B = \ell^\infty(\mathcal{F})$  where  $\mathcal{F}$  is a class of real - valued functions defined on a probability space  $(\mathcal{X}, \mathcal{A}, P)$
- $\ell^\infty(\mathcal{F}) = \{\phi : \mathcal{F} \rightarrow \mathbb{R} \mid \|\phi\|_{\mathcal{F}} \equiv \sup_{f \in \mathcal{F}} |\phi(f)| < \infty\};$
- $Z_i = \delta_{X_i} - P, \quad Z_i(f) = f(X_i) - Pf, \text{ for } i \in \{1, \dots, n\};$
- $X_1, \dots, X_n$  are i.i.d.  $P$ ;
- $S_n(f) = n^{-1/2} \sum_{i=1}^n (f(X_i) - Pf) \equiv \mathbb{G}_n(f);$   
 $S_n^\xi(f) = n^{-1/2} \sum_{i=1}^n \xi_i(f(X_i) - Pf) \equiv \mathbb{G}_n^\xi(f);$
- $\mathbb{P}_n = n^{-1} \sum_{i=1}^n \delta_{X_i};$
- $\|g\|_n^2 \equiv \mathbb{P}_n g^2 = n^{-1} \sum_{i=1}^n g^2(X_i).$

**Method for proving A3:** The classical multiplier inequality.

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**Classical Multiplier Inequality:** Pisier; Giné and Zinn (1984).

Let  $Z_1, \dots, Z_n$  be i.i.d. with  $E\|Z\|_{\mathcal{F}}^2 < \infty$ , independent of

$\epsilon_1, \dots, \epsilon_n$  i.i.d. Rademacher rv's.

Then for every i.i.d. sample  $\xi_1, \dots, \xi_n$  of zero mean rv's independent of  $Z_1, \dots, Z_n$  and any  $1 \leq n_0 \leq n$ ,

$$\begin{aligned} & \frac{1}{2} \|\xi_1\|_1 E \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i Z_i \right\|_{\mathcal{F}} \\ & \leq E \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i Z_i \right\|_{\mathcal{F}} \\ & \leq 2(n_0 - 1) E\|Z_1\|_{\mathcal{F}} \cdot E \max_{1 \leq i \leq n} \frac{|\xi_i|}{\sqrt{n}} \\ & \quad + 2\sqrt{2} \|\xi_1\|_{2,1} \max_{n_0 \leq k \leq n} E \left\| \frac{1}{\sqrt{k}} \sum_{i=n_0}^k \epsilon_i Z_i \right\|_{\mathcal{F}}. \end{aligned}$$

### 3. A new multiplier inequality

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**Assumptions:**  $\xi_1, \dots, \xi_n$  are independent of  $X_1, \dots, X_n$   
and either:

**A1**  $X_1, \dots, X_n$  are i.i.d.  $P$  on  $(\mathcal{X}, \mathcal{A})$  and  $\mathcal{F}$  is  $P$ -centered,  
 $Pf = \int f dP = 0$  for all  $f \in \mathcal{F}$ ,

or

**A2**  $X_1, \dots, X_n$  are permutation invariant and  $\xi_1, \dots, \xi_n$  are  
independent mean 0 random variables.

**Entropy Conditions F:** for some  $0 < \alpha < 2$

**F1:**  $\log N_{[]}(\epsilon, \mathcal{F}, L_2(P)) \lesssim \epsilon^{-\alpha}$ .

**F2:**  $\sup_Q \log N(\epsilon \|F\|_{L_2(Q)}, \mathcal{F}, L_2(Q)) \lesssim \epsilon^{-\alpha}$ .

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**Theorem. (Han and W, 2019)** Suppose Assumption A holds. Let  $\{\mathcal{F}\}_{k=1}^n$  be a sequence of function classes such that  $\mathcal{F}_k \supset \mathcal{F}_n$  for all  $1 \leq k \leq n$ . Suppose that

$$E \left\| \sum_{i=1}^k \epsilon_i f(X_i) \right\|_{\mathcal{F}_k} \leq \psi_n(k), \quad 1 \leq k \leq n$$

for nondecreasing concave functions  $\{\psi_n\} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  with  $\psi_n(0) = 0$ . Then:

$$E \left\| \sum_{i=1}^n \xi_i f(X_i) \right\|_{\mathcal{F}_n} \leq 4 \int_0^\infty \psi_n \left( \sum_{i=1}^n P(|\xi_i| \geq t) \right) dt. \quad (4)$$

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**Corollary.** Suppose that the assumptions of Theorem 1 hold and that the marginal distributions of the  $\{\xi_i\}$  are all equal. Suppose that  $\psi_n(k) = \kappa_0 k^{1/\gamma}$  for all  $1 \leq k \leq n$  and for some  $\gamma \geq 1$  and  $\kappa_0 > 0$ . Then for any  $p \geq 1$  such that  $\|\xi_1\|_{p,1} < \infty$ ,

$$E \left\| \sum_{i=1}^n \xi_i f(X_i) \right\|_{\mathcal{F}_n} \leq 4\kappa_0 n^{\max\{1/\gamma, 1/p\}} \cdot \|\xi_1\|_{\gamma \wedge p, 1}.$$

**Remark 1:** The right side of (4) depends only on the marginal distributions of the  $\xi_i$ 's. Independence of the coordinates of  $\xi \equiv (\xi_1, \dots, \xi_n)$  has not been assumed under **A1**.

**Remark 2:** The functions  $\psi_n$  measure the “size” of the usual empirical processes.

**Remark 3:** The multiplier CLT is still a corollary.

**Remark 4:** Example of a setting in which the new inequality is strictly better.

**Remark 5:** Witness theorem type lower bound.

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**Remark 4:** The new inequality can be strictly better.

Let the  $X_i$ 's be i.i.d.  $\text{Uniform}(0,1)$ , and let  $\mathcal{F}$  be a class of bounded functions on  $[0,1]$  satisfying the entropy condition  $F$  with  $0 < \alpha < 2$ . Let  $\xi_1, \dots, \xi_n$  be i.i.d. with  $\|\xi_1\|_{p,1} < \infty$  with  $p \geq 2$ . Let  $\mathcal{F}_n \equiv \mathcal{F}(\delta_n)$  where

$$\mathcal{F}(\delta) \equiv \{f \in \mathcal{F} : Pf^2 \leq \delta^2\}, \quad \delta_n \equiv n^{-1/(2+\alpha)}.$$

Then the classical multiplier inequality yields

$$\begin{aligned} E \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i f(X_i) \right\|_{\mathcal{F}(\delta_n)} \\ &\lesssim \inf_{1 \leq n_0 \leq n} \left\{ \frac{n_0}{n^{1/2-1/p}} + n_0^{-(2-\alpha)/(2(2+\alpha))} \right\} \\ &\asymp n^{-\beta_1} \quad \text{with} \quad \beta_1 \equiv \frac{2-\alpha}{6+\alpha} \left( \frac{1}{2} - \frac{1}{p} \right). \end{aligned}$$

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The new multiplier inequality yields

$$E \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i f(X_i) \right\|_{\mathcal{F}(\delta_n)} \lesssim n^{-\min\{\frac{2-\alpha}{2(2+\alpha)}, \frac{1}{2} - \frac{1}{p}\}} \equiv n^{-\beta_2} \quad (5)$$

where

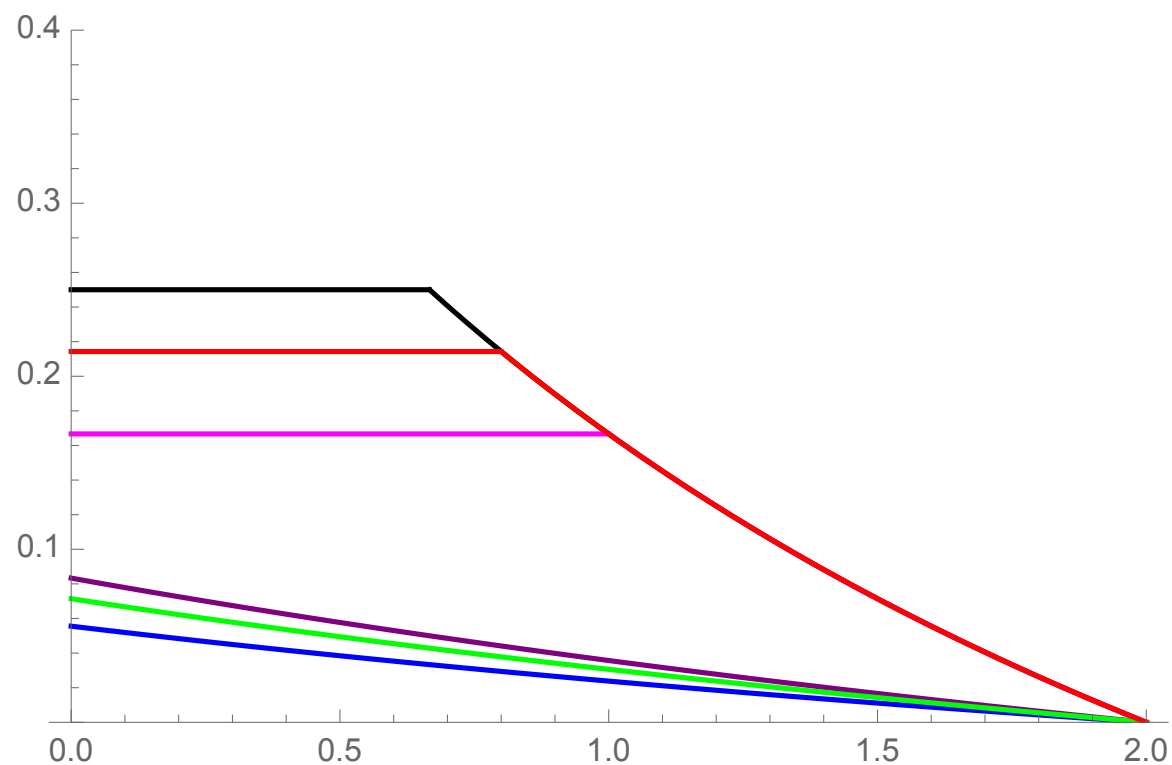
$$\begin{aligned} \beta_1 &= \frac{2-\alpha}{6+\alpha} \left( \frac{1}{2} - \frac{1}{p} \right) \\ &< \min \left\{ \frac{2-\alpha}{2(2+\alpha)}, \frac{1}{2} - \frac{1}{p} \right\} = \beta_2. \end{aligned}$$

When  $p \geq 1 + 2/\alpha$ , the bound in (5) becomes

$$n^{-\frac{2-\alpha}{2(2+\alpha)}}$$

which matches the rate for the symmetrized empirical process:

$$E \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i f(X_i) \right\|_{\mathcal{F}(\delta_n)} \lesssim n^{-(2-\alpha)/(2(2+\alpha))}.$$



$\beta_1(\alpha, p)$  for  $0 < \alpha \leq 2$ ,  $p \in \{3, 3.5, 4\}$ , (blue, green, purple)  
 $\beta_2(\alpha, p)$  for  $0 < \alpha \leq 2$ ,  $p \in \{3, 3.5, 4\}$ , (magenta, red, black)

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**Remark 5:** (Witness lower bound).

**Theorem 2.** Let  $\mathcal{X} = [0, 1]$  and let  $P$  be a probability measure on  $\mathcal{X}$  with Lebesgue density  $h$  satisfying  $c^{-1} \leq \inf_{0 < x < 1} h(x) \leq \sup_{0 < x < 1} h(x) \leq c$  for some  $0 < c < \infty$ . Let  $\xi_1, \dots, \xi_n$  be i.i.d random variables such that

$$E \max_{1 \leq i \leq n} |\xi_i| \geq \kappa_0 n^{1/p} \quad \left( P(|\xi_1| > t) \geq \kappa'_0 / (1 + t^p), \quad t > 0 \text{ works} \right)$$

for some  $1 < p < \infty$  and  $\kappa_0 > 0$ . Then for any  $\gamma > 2$  there exists a sequence  $\{\mathcal{F}_k\}_{k=1}^n$  defined on  $\mathcal{X}$  with  $\mathcal{F}_k \supset \mathcal{F}_n$  for  $1 \leq k \leq n$  such that for  $n$  sufficiently large

$$E \left\| \sum_{i=1}^k \epsilon_i f(X_i) \right\|_{\mathcal{F}_k} \leq \kappa_1 k^{1/\gamma} \quad \text{for all } 1 \leq k \leq n$$

and

$$E \left\| \sum_{i=1}^n \xi_i f(X_i) \right\|_{\mathcal{F}_n} \geq \frac{1}{\kappa_1} n^{1/\gamma \vee 1/p}.$$

where  $\kappa_1 \equiv \kappa_1(\kappa_0, \gamma, p)$ .

## 4. Consequences for Least Squares Regression

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**Theorem 1. (Least Squares Rate)** Suppose that  $\xi_1, \dots, \xi_n$  are i.i.d. errors with  $E\xi_1 = 0$  and  $\|\xi_1\|_{p,1} < \infty$  for some  $p \geq 1$ . Suppose  $\mathcal{F}$  is a  $P$ -centered function class with  $\mathcal{F} - f_0 \subset L_\infty(1)$  satisfying the entropy condition (F) for some  $0 < \alpha < 2$ . Then the LSE  $\hat{f}_n$  satisfies

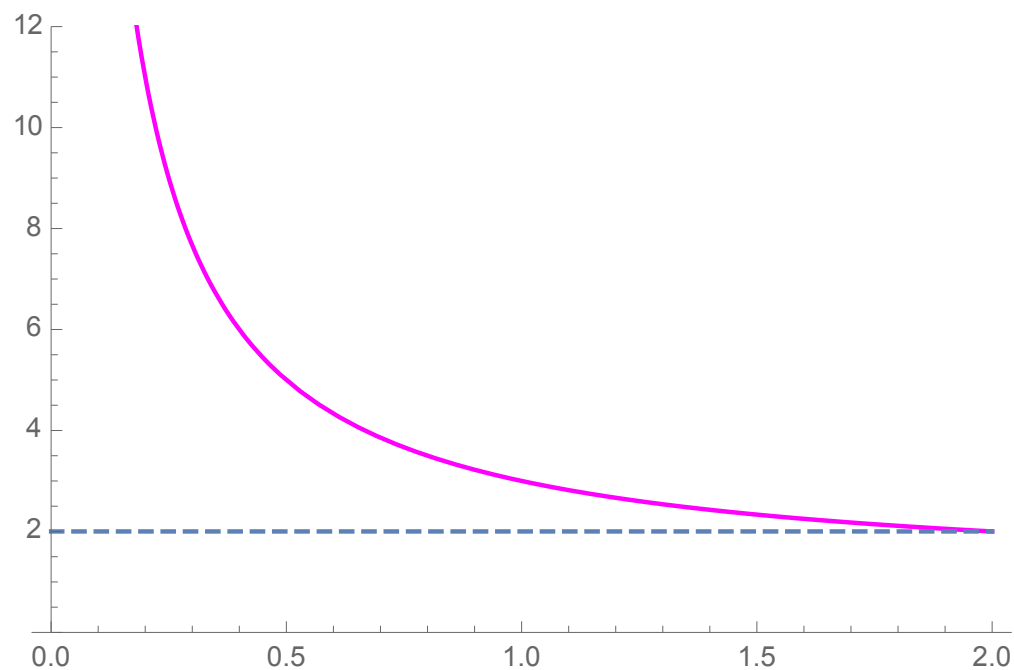
$$\|\hat{f}_n - f_0\|_{L_2(P)} = O_p \left( n^{-1/(2+\alpha)} \vee n^{-\frac{1}{2} + \frac{1}{2p}} \right). \quad (6)$$

If  $p \geq 2$ , then (6) holds in expectation:

$$E\|\hat{f}_n - f_0\|_{L_2(P)} = O \left( n^{-1/(2+\alpha)} \vee n^{-\frac{1}{2} + \frac{1}{2p}} \right).$$

**Corollary:** Suppose the assumptions of Theorem 1 hold with  $p \in (1, 2]$ . Then

$$\|\hat{f}_n - f_0\|_{L_2(P)} = O_p \left( n^{-\frac{1}{2} + \frac{1}{2p}} \right) = o_p(1).$$



Plot of  $p_{crit}(\alpha) = 1 + 1/(2\alpha)$

Above this curve: the minimax rate  $n^{-1/(2+\alpha)}$  holds;

Below this curve: the rate  $n^{-(1/2-1/(2p))}$  is driven by the  $p$  moment of the  $\xi_i$ 's

- 
- If  $\alpha = 1$  (so 1-smooth functions in  $d = 1$ ), if  $p \geq 3$ , the Gaussian rate  $n^{-1/3}$  holds.
  - If  $\alpha = 1/2$  (so 2-smooth functions in  $d = 1$ ), if  $p \geq 5$ , the Gaussian rate  $n^{-2/5}$  holds.

## 5. A consequence for the Lasso

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Suppose that

$$\mathbf{Y} = \mathbf{X}\theta_0 + \xi$$

where

- $\mathbf{Y} \in \mathbb{R}^n$
- $\mathbf{X} \in \mathbb{R}^{n \times d}$  is a random design matrix;
- $\theta_0 \in \mathbb{R}^d$  is a vector of regression coefficients;
- $\xi = (\xi_1, \dots, \xi_n)$  is a mean-zero noise vector independent of  $\mathbf{X}$ .

When  $\theta_0 \in \mathbb{R}^d$  is sparse, one popular estimator is the Lasso:

$$\hat{\theta}(\lambda) = \operatorname{argmin}_{\theta \in \mathbb{R}^d} \left( \frac{1}{n} \|\mathbf{Y} - \mathbf{X}\theta\|_2^2 + \lambda \|\theta\|_1 \right)$$

where  $\lambda$  is typically chosen via cross-validation.

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**Q5** Under what moment assumptions on  $\mathbf{X}$  and on the noise  $\xi$  can the lasso estimator achieve the optimal rate of convergence?

Most work on this assumes that both  $\mathbf{X}$  and  $\xi$  are sub-Gaussian:  $Ee^{\alpha\xi_1^2} < \infty$  and similarly for the components (or individual entries) of  $\mathbf{X}$ . To state our results relaxing these moment conditions, we will need several further definitions and bits of notation.

**Compatibility condition:** For  $L > 0$  and  $S \subset \{1, \dots, d\}$ , let

$$\phi(L, S) = \sqrt{|S|} \min \left\{ \frac{1}{\sqrt{n}} \|\mathbf{X}\theta_S - \mathbf{X}\theta_{S^c}\|_2 : \|\theta_S\|_1, \|\theta_{S^c}\|_1 \leq L \right\}$$

Here for  $\theta = (\theta_i) \in \mathbb{R}^d$ ,  $\theta_S = (\theta_i 1_S(i))$ , and  $\theta_{S^c} = (\theta_i 1_{S^c}(i))$

- 
- Let  $B_0(s) = \{\theta \in \mathbb{R}^d : |\{i : \theta_i \neq 0\}| \leq s\}$ .
  - Let  $\Sigma = E(\widehat{\Sigma})$  where  $\widehat{\Sigma} \equiv \mathbf{X}^T \mathbf{X} / n =$  the sample covariance matrix.
  - $\underline{\sigma}_d = \sigma_{\min}(\Sigma)$ ,  $\bar{\sigma}_d(\Sigma) = \sigma_{\max}(\Sigma)$  be the smallest and largest singular values of  $\Sigma$ . Here  $d = d_n$ ,  $s = s_n$ .

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**Theorem.** Let  $\mathbf{X}$  be a design matrix with i.i.d. mean-zero rows and suppose that

$$0 < \liminf \underline{\sigma}_d \leq \limsup \bar{\sigma}_d < \infty.$$

Suppose that  $\min_{S: |S| \leq s} \phi(3, S) \geq c_0$  holds for some  $c_0 > 0$  with probability converging to 1 as  $n \rightarrow \infty$ . Suppose as well that

$$\limsup_{n \rightarrow \infty} \frac{\log(d) (M_4(X) \vee \log^2 d)}{n^{2-4\alpha}} < \infty$$

where  $M_4(\mathbf{X}) = E\{\max_{1 \leq j \leq d} |X_{1,j}|^4\}$ . Then for

$$\hat{\theta}^L \equiv \hat{\theta} \left( 2L \|\xi_n\|_{1/\alpha,1} \sqrt{(\log d)/n} \right)$$

$$Q_n(s, L) \equiv \sup_{\theta_0 \in B_0(s)} P_{\theta_0} \left( \frac{1}{n} \|\mathbf{X}(\hat{\theta}^L - \theta_0)\|_2^2 > \frac{16L^2 \|\xi_n\|_{1/\alpha,1}}{c_0^2} \cdot \frac{s \log d}{n} \right)$$

satisfies

$$\lim_{L \nearrow \infty} \overline{\lim}_{n \rightarrow \infty} Q_n(s, L) = 0.$$

## 6. Further consequences & further problems

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- A consequence for the lasso with heavy-tailed  $\xi_i$ 's and  $X_i$ 's.
- A result for LS estimators under an envelope condition.
- Additive regression under shape restrictions
- ?? Trans-Donsker classes with  $\alpha > 2$ ?
- ?? Convex LS regression with  $d > 4$ ?

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## Some references:

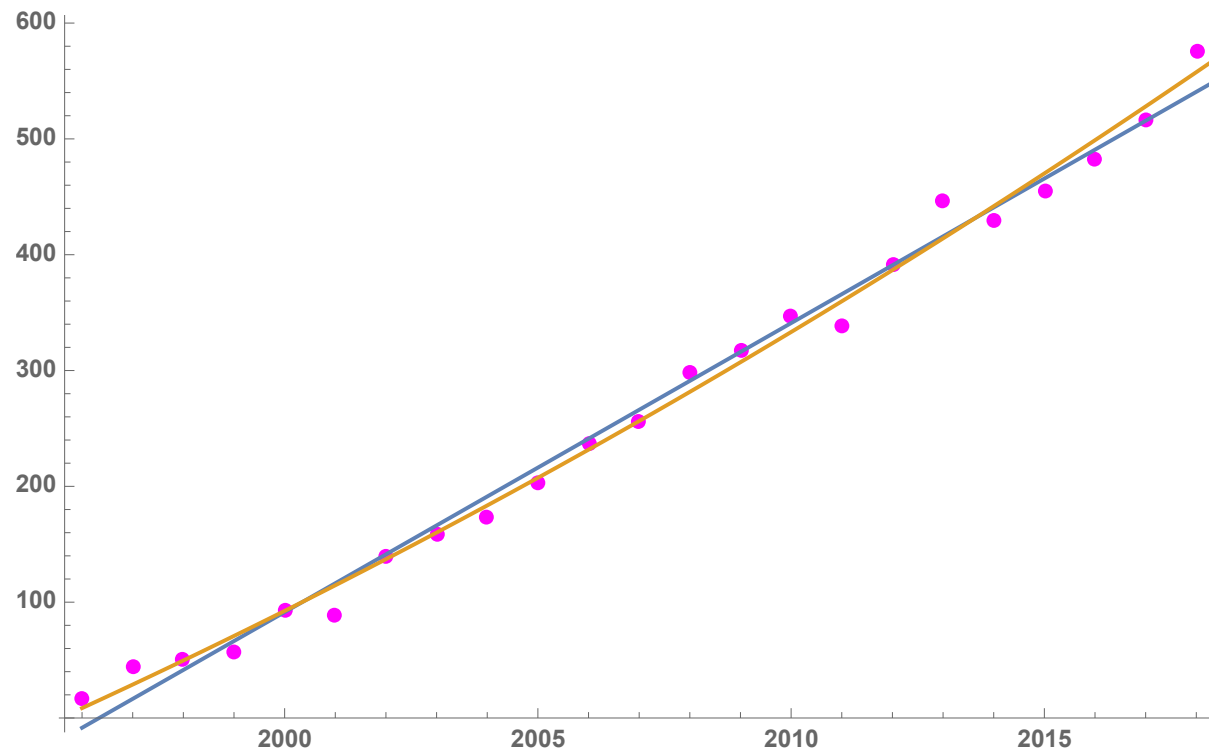
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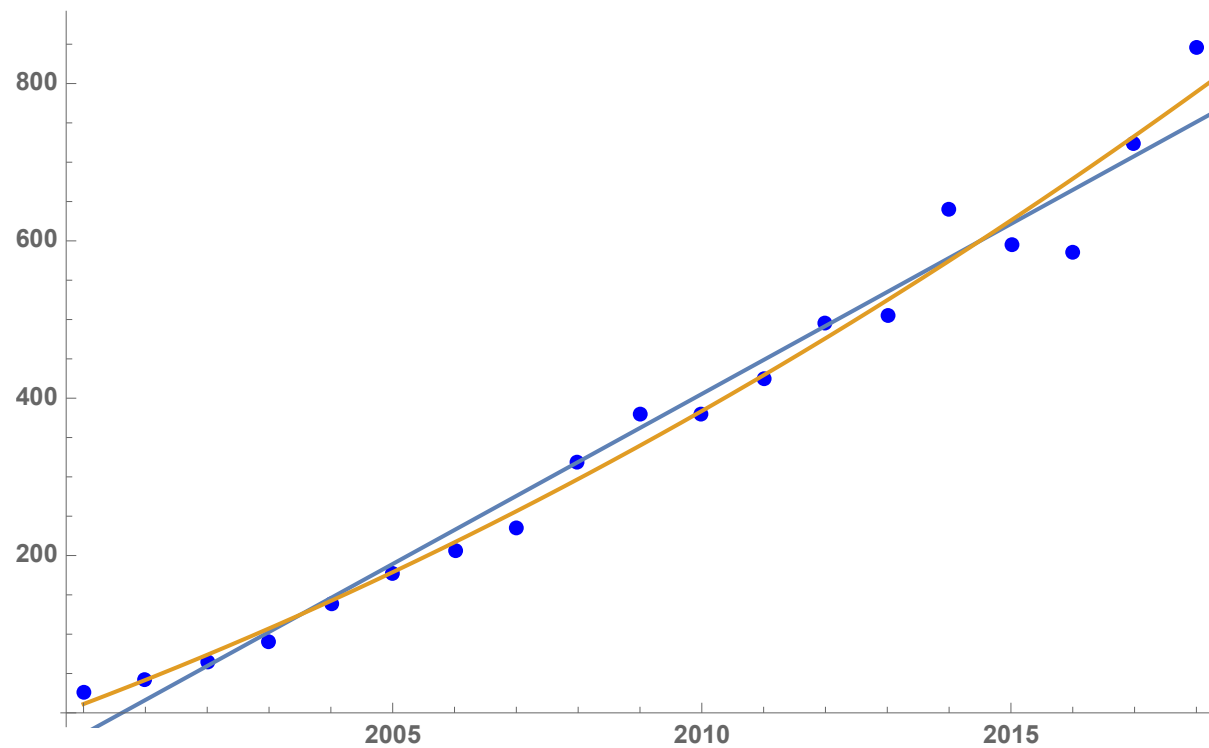
Aad and Jon, Hortus Botanicus, 2015

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HAPPY BIRTHDAY  
Aad!



Cites of WCES: linear and quadratic fits



Cites of AS: linear and quadratic fits