

How to estimate a density on a spider web ?

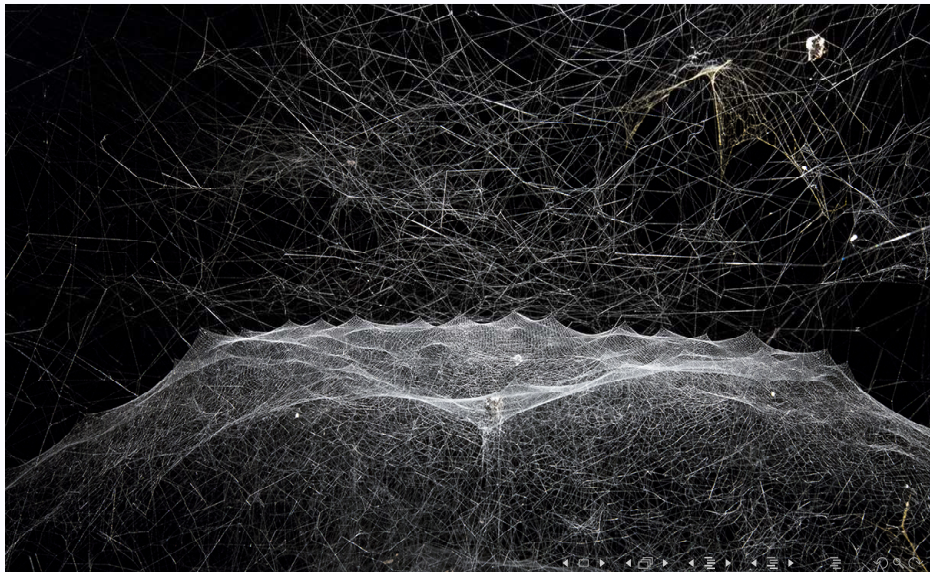
Dominique Picard



BON ANNIVERSAIRE AAD



SPIDER WEB FROM EXHIBITION ON AIR TOMAS SARACENO



SPIDER WEB



SPIDER WEB



SPIDER

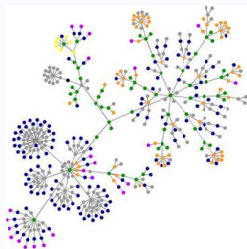


DENSITY ESTIMATION PROBLEM

- one observes $X_1, \dots, X_n$ that are i.i.d. random variables defined on a space $\mathcal{M}$ with the common density f .
- $\mathcal{M}$ is equipped with a distance ρ and a measure μ .
- our aim is to identify the density

$\mathcal{M}$ = SPIDER WEB

We identify $\mathcal{M}$ with a graph (simple, undirected, no loops) with T vertices and edges.



Adjacency matrix $A_{ij} = 1$ if there is an edge between i and j , 0, otherwise.

$\mathcal{M}$ is naturally equipped with a distance (geodesic-distance)

$$\rho(x, y) = \inf_{\text{paths } \gamma \subset \mathcal{M}, \gamma(0)=x, \gamma(L)=y} \sum_{i=0}^{L-1} A_{\gamma(i)\gamma(i+1)}.$$

GRAPH LAPLACIAN

we calculate a Laplacian matrix $L_{T \times T}$ as:

$$L = D - A,$$

where D is the (diagonal) degree matrix (popularity of each vertex) and A is the adjacency matrix of the graph.

$$D_{ii} = \sum_{j \neq i} A_{i,j}$$

is the degree of the vertex i .

SPECTRAL DECOMPOSITION OF L

L has $\lambda_1 \geq \dots \geq \lambda_T$ for eigenvalues and $V^1, \dots, V^T$ for eigenvectors (normed) : $V^j = (V_1^j, \dots, V_T^j)$.

Our estimator is : for x point of the graph :

$$\hat{K}_\delta(x) := \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^T \Phi(\delta \sqrt{\lambda_j}) V_{X_i}^j V_x^j$$

δ is the bandwidth, Φ is a '**Littlewood-Paley function**' : Φ be an even $C^\infty(\mathbb{R})$ real-valued function with the following properties:

$$\text{supp}(\Phi) \subset [-1, 1] \quad \text{and} \quad \Phi(\lambda) = 1 \quad \text{for } \lambda \in [-1/2, 1/2].$$

Theoretical part : Kernel and wavelet density estimators on manifolds and more general metric spaces

Issued from a joint work :

G. Cleanthous, A. Georgiadis, G. Kerkyacharian, P. Petrushev, D. P.

SETTING AND MOTIVATION

We assume that $(\mathcal{M}, \rho, \mu)$ is a metric measure space equipped with a distance ρ and a measure μ .

$X_1, \dots, X_n$ are i.i.d. random variables on $\mathcal{M}$ with a common density function (pdf) f with respect to the measure μ . Our purpose is to estimate the density f .

To an estimator $\hat{f}_n$ of f , we associate its risk:

$$R_n(\hat{f}, f, p) = \mathbb{E}_f \left(\int_{\mathcal{M}} |\hat{f}_n(x) - f(x)|^p \mu(dx) \right)^{\frac{1}{p}} \quad 1 \leq p < \infty$$

as well as its $\mathbb{L}_\infty$ risk:

$$R_n(\hat{f}, f, \infty) = \mathbb{E}_f \|\hat{f}_n - f\|_\infty.$$

We will operate in the following setting. Most of the material can be found in an extended form in the papers [4, 8] [Coulhon, Kerkycharian, Petrushev](#).

DOUBLING CONDITION : SETTING THE DIMENSION

C1. We assume that the metric space $(\mathcal{M}, \rho, \mu)$ satisfies the so called *doubling volume condition*:

$$\mu(B(x, 2r)) \leq c_0 \mu(B(x, r)) \quad \text{for } x \in \mathcal{M} \text{ and } r > 0, \quad (1)$$

where $B(x, r) := \{y \in \mathcal{M} : \rho(x, y) < r\}$ and $c_0 > 1$ is a constant. The above implies that there exist constants $c'_0 \geq 1$ and $d > 0$ such that

$$\mu(B(x, \lambda r)) \leq c'_0 \lambda^d \mu(B(x, r)) \quad \text{for } x \in \mathcal{M}, r > 0, \text{ and } \lambda > 1, \quad (2)$$

The least d such that (2) holds is the so called *homogeneous dimension* of $(\mathcal{M}, \rho, \mu)$

We can additionally assume that $(\mathcal{M}, \rho, \mu)$ is a compact measure space with $\mu(\mathcal{M}) < \infty$ satisfying the following condition:

C1A. *Ahlfors regular volume condition:* There exist constants $c_1, c_2 > 0$ and $d > 0$ such that

$$c_1 r^d \leq |B(x, r)| \leq c_2 r^d \quad \text{for } x \in \mathcal{M} \text{ and } 0 < r \leq \text{diam}(\mathcal{M}).$$

The doubling condition is precisely related to the *metric entropy* : For $\epsilon > 0$, we define, as usual, the covering number $N(\epsilon, \mathcal{M})$ as the smallest number of balls of radius ϵ covering $\mathcal{M}$.

Lemma

*Under the condition **C1A** and if $\mathcal{M}$ is compact, there exist constants $c' > 0$, $c'' > 0$ and $\epsilon_0 > 0$ such that*

$$\frac{1}{c'} \left(\frac{1}{\epsilon} \right)^d \leq N(\epsilon, \mathcal{M}) \leq \frac{2^d}{c''} \left(\frac{1}{\epsilon} \right)^d,$$

for all $0 < \epsilon \leq \epsilon_0$.

SMOOTH OPERATOR : SETTING REGULARITY, CONSTRUCT A KERNEL

One rather standard method in density estimation is the kernel estimation method, i.e. considering a family of functions indexed by $\delta > 0$:

$K_\delta : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ an associated kernel density estimator is defined by

$$\hat{K}_\delta(x) := \frac{1}{n} \sum_{i=1}^n K_\delta(X_i, x), \quad x \in \mathcal{M}. \quad (3)$$

In $\mathbb{R}^d$, an important family is the family of translation kernels

$K_\delta(x, y) = [\frac{1}{\delta}]^d G(\frac{x-y}{\delta})$, where G is a function $\mathbb{R}^d \rightarrow \mathbb{R}$. When $\mathcal{M}$ is a more involved set such as a spider web, a manifold or a set of graphs, of matrices, **the simple operations of translation and dilation may not be meaningful**. Hence, even finding a family of kernels to start with might be a difficulty.

When dealing with a kernel estimation method, it is standard to consider two quantities :

$$b_\delta(f) := \|\mathbb{E}_f \widehat{K}_\delta - f\|_p, \quad \|\xi_f\|_p := \|\widehat{K}_\delta - \mathbb{E}_f \widehat{K}_\delta\|_p.$$

The analysis of the first term $b_\delta(f)$ is linked to the approximation properties of the family $\mathbb{E}_f \widehat{K}_\delta$.

which are also often linked to regularity properties of the function f . It is standardly proved (see e.g. [7]), that in $\mathbb{R}^d$ if K is a **translation family with mild properties on K** , then **polynomial rates of approximation are obtained for functions with Besov regularity**.

REGULARITY SPACES

Hence, an important issue becomes finding spaces of regularity associated to a possibly complex set $\mathcal{M}$. On a compact metric space (M, ρ) one can always define the scale of s -Lipschitz spaces defined by the following norm

$$\|f\|_{Lip_s} := \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^s}, \quad 0 < s \leq 1. \quad (4)$$

In Euclidian spaces a function can be much **more regular than Lipschitz**, for instance differentiable at different orders, or belong to some Sobolev or Besov spaces.

When $\mathcal{M}$ is a set where there is **no obvious notion of differentiability**, one can make the observation that in $\mathbb{R}^d$ or **Riemannian manifolds**, **regularity properties can also be expressed via the associated Laplacian**. The Laplacian itself is an operator of order 2, but its square root is of order 1 and can be interpreted as a substitute for derivation.

SMOOTH OPERATOR

Our main assumption is that the space $(\mathcal{M}, \rho, \mu)$ is complemented by an essentially self-adjoint non-negative operator L on $\mathbb{L}^2(\mathcal{M}, \mu)$, mapping real-valued to real-valued functions, such that the associated semigroup $P_t = e^{-tL}$ consists of integral operators with the (heat) kernel $p_t(x, y)$ obeying the conditions mentioned in the sequel.

$$p_t(x, y) := \sum_k e^{-t\lambda_k} P_k(x, y),$$

with $P_k(x, y) = \sum_i v_i^{\lambda_k}(x) \overline{v_i^{\lambda_k}(y)}$, where $v_i^{\lambda_k}(x), i = 1, \dots, \dim(E_{\lambda_k})$ is an orthonormal basis of E_{λ_k} , when the spectrum is discrete.

C3. Gaussian localization: There exist constants $c_4, c_5 > 0$ such that

$$|p_t(x, y)| \leq \frac{c_4 \exp\left(-\frac{c_5 \rho^2(x, y)}{t}\right)}{(|B(x, \sqrt{t})| |B(y, \sqrt{t})|)^{1/2}} \quad \text{for } x, y \in \mathcal{M}, t > 0. \quad (5)$$

C4. Hölder continuity: There exists a constant $\alpha > 0$ such that

$$|p_t(x, y) - p_t(x, y')| \leq c_4 \left(\frac{\rho(y, y')}{\sqrt{t}}\right)^\alpha \frac{\exp\left(-\frac{c_5 \rho^2(x, y)}{t}\right)}{(|B(x, \sqrt{t})| |B(y, \sqrt{t})|)^{1/2}} \quad (6)$$

for $x, y, y' \in \mathcal{M}$ and $t > 0$, whenever $\rho(y, y') \leq \sqrt{t}$.

C5. Markov property:

$$\int_{\mathcal{M}} p_t(x, y) d\mu(y) = 1 \quad \text{for } x \in \mathcal{M} \text{ and } t > 0. \quad (7)$$

ASSOCIATED REGULARITY SPACES

Regularity classes defined through approximation properties:

$$\Sigma_u = \bigoplus_{\sqrt{\lambda_k} \leq u} \mathcal{H}_{\lambda_k}$$

$$B_p^s = \{f \in \mathbb{L}^p, \quad \inf_{\phi \in \Sigma_u} \|f - \phi\|_p \leq Cu^{-s}\}$$

(usually denoted $B_{p,\infty}^s$). Moreover, one can develop a theory of "wavelet" which completely characterizes the previous regularity classes.

TYPICAL EXAMPLES $\mathcal{M} = \mathbb{R}^d$

Here $d\mu$ is the Lebesgue measure and ρ is the Euclidean distance on $\mathbb{R}^d$. In this case we consider the operator

$$-L(f)(x) = \sum_{j=1}^d \partial_j^2 f(x) = \operatorname{div}(\nabla f)(x)$$

defined on the space $\mathcal{D}(\mathbb{R}^d)$ of C^∞ functions with compact support. As is well known the operator L is positive essentially self-adjoint. The associate semigroup $e^{t\Delta}$ is given by the operator with the Gaussian kernel:

$$p_t(x, y) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

PERIODIC CASE ON $\mathcal{M} = [-1, 1]$

Here $d\mu$ is the Lebesgue measure and ρ is the Euclidean distance on the circle. The operator is $L(f) = -f''$ defined on the set on infinitely differentiable periodic functions. It has eigenvalues $k^2\pi^2$ for $k \in \mathbb{N}_0$ and eigenspaces

$$\ker(L) = \mathcal{H}_0 = \text{span}\left\{\frac{1}{\sqrt{2}}\right\}, \quad \ker(L - k^2\pi^2) = \mathcal{H}_k = \text{span}\{\cos k\pi x, \sin k\pi x\}$$

UNIT SPHERE $\mathcal{M} = \mathbb{S}^{d-1}$ IN $\mathbb{R}^d$, $d \geq 3$

This is the most famous Riemannian manifold with the induced structure from $\mathbb{R}^d$. Here $d\mu$ is the Lebesgue measure on $\mathbb{S}^{d-1}$, ρ is the geodesic distance on $\mathbb{S}^{d-1}$:

$$\rho(\xi, \eta) = \arccos(\langle \xi, \eta \rangle_{\mathbb{R}^d}),$$

and $L := -\Delta_0$ with Δ_0 being the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. The spectral decomposition of the operator L can be described as follows:

$$\mathbb{L}^2(\mathbb{S}^{d-1}, \mu) = \bigoplus E_{\lambda_k}, \quad E_{\lambda_k} = \ker(L - \lambda_k I_d), \quad \lambda_k = k(k + d - 2).$$

Here E_{λ_k} is the restriction to $\mathbb{S}^{d-1}$ of harmonic homogeneous polynomials of degree k (spherical harmonics). We have $\dim(E_{\lambda_k}) = \binom{d-1}{d+k-1} - \binom{d-1}{d+k-3}$.

LIE GROUP OF MATRICES: $\mathcal{M} = SU(2)$

This example is interesting in astrophysical problems, especially in the measures associated to the CMB, where instead of only measuring the intensity of the radiation we also measure spins. By definition

$$SU(2) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

Thus

$$q \in SU(2) \leftrightarrow q \in M(2, \mathbb{C}), \quad q^{-1} = -q^*, \quad \det(q) = 1.$$

This a compact group which topologically is the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$.

$$\rho_{SU(2)}(x, y) = \arccos \frac{1}{2} \text{Tr}[xy^*]$$

The eigenvalues of $L = -\Delta$ are $\lambda_k = k(k+2)$ and the dimension of the respective eigenspaces E_{λ_k} is $(k+1)^2$.

CONSTRUCTION OF KERNEL DENSITY ESTIMATORS ON THE METRIC MEASURE SPACE $\mathcal{M}$

To explain our construction of kernel estimators we begin by considering the classical example of the periodic case on $\mathcal{M} = [-1, 1]$. A commonly admitted nonparametric estimator is of the form

$$\hat{f}_T(x) = \frac{1}{2} + \frac{1}{n} \sum_{i=1}^n \sum_{1 \leq k \leq T} \cos k\pi(x - X_i).$$

It falls into the category of orthogonal series estimator. It is well known that these estimators have nice $\mathbb{L}^2$ properties but can drastically fail in $\mathbb{L}^p$, $p \neq 2$, or locally.

In our setting we will replace $\hat{f}_T$ by a 'smoothed version':

$$\hat{\Phi}_\delta(x) = \frac{1}{2} + \frac{1}{n} \sum_{i=1}^n \sum_{k \geq 1} \Phi(\delta k) \cos k\pi(x - X_i) =: \frac{1}{n} \sum_{i=1}^n \Phi_\delta(x, X_i),$$

where Φ is Littlewood Paley function.

In analogy to this case, replacing the circle by $\mathcal{M}$ and the Laplacian by the operator $-L$

Definition

Let $X_1, \dots, X_n$ be i.i.d. random variables on $\mathcal{M}$ in our setting.

$$\hat{\Phi}_\delta(x) := \frac{1}{n} \sum_{i=1}^n \Phi(\delta\sqrt{L})(X_i, x), \quad x \in \mathcal{M}.$$

$$\Phi(\delta\sqrt{L})(x, y) := \int_0^\infty \Phi(\delta\sqrt{\lambda}) dE_\lambda(x, y) = \sum_k \Phi(\delta\sqrt{\lambda_k}) P_k(x, y)$$

with $P_k(x, y) = \sum_i v_i^{\lambda_k}(x) \overline{v_i^{\lambda_k}(y)}$, where $v_i^{\lambda_k}(x), i = 1, \dots, \dim(E_{\lambda_k})$ is an orthonormal basis of E_{λ_k} , when the spectrum is discrete.

Φ is a 'Littlewood-Paley' function.

EXAMPLES OF LITTLEWOOD PALEY KERNEL ESTIMATES

- For the sphere, we get the following estimator

$$\widehat{\Phi}_\delta(x) = \frac{1}{n} \sum_{i=1}^n \sum_k \Phi(\delta \sqrt{k(k+d-2)}) L_k(\langle x, X_i \rangle_{\mathbb{R}^d}), \quad \forall x \in \mathbb{S}^{d-1}$$

where

$$L_k(x) = \frac{1}{|\mathbb{S}^{d-1}|} \left(1 + \frac{k}{\nu}\right) C_k^\nu(x), \quad \nu = \frac{d-2}{2}$$

- For $SU(2)$, we get the following estimator,

$$\widehat{\Phi}_\delta(x) = \frac{1}{n} \sum_{i=1}^n \sum_k \Phi(\delta \sqrt{k(k+2)}) L_k\left(\frac{1}{2} \text{Tr}[X_i x^*]\right), \quad \forall x \in SU(2)$$

with $L_k(x) = \frac{1}{|\mathbb{S}^3|} (1+k) C_k^1(x)$.

UPPER BOUND RESULTS

Theorem

(i) If $2 \leq p < \infty$, then

$$\mathbb{E} \|\widehat{\Phi}_\delta - f\|_p \leq \frac{c(p, f)}{(n\delta^d)^{\frac{1}{2}}} + \|\Phi(\delta\sqrt{L})f - f\|_p, \quad 0 < \delta \leq 1.$$

(ii) If $1 \leq p < 2$ and $\text{supp}(f) \subset B(x_0, R)$ for some $x_0 \in \mathcal{M}$ and $R > 0$, then

$$\mathbb{E} \|\widehat{\Phi}_\delta - f\|_p \leq \frac{c(p)}{(n\delta^d)^{\frac{1}{2}}} |B(x_0, R)|^{\frac{1}{p} - \frac{1}{2}} + \|\Phi(\delta\sqrt{L})f - f\|_p, \quad 0 < \delta \leq 1.$$

(iii) There exists a constant c such that for any $q \geq 2$ and $0 < \delta \leq 1$ we have

$$\mathbb{E} \|\widehat{\Phi}_\delta - f\|_\infty \leq c\delta^{-\frac{d}{q}} \left(\frac{q}{(n\delta^d)^{1-\frac{1}{q}}} + \frac{q^{1/2}}{(n\delta^d)^{\frac{1}{2}}} \|f\|_\infty^{\frac{1}{2}-\frac{1}{q}} \right) + \|\Phi(\delta\sqrt{L})f - f\|_\infty.$$

MINIMAX-TYPE RESULTS

Theorem

(i) If $2 \leq p < \infty$ and $\delta = n^{-\frac{1}{2s+d}}$, then

$$\sup_{f \in B_{p\tau}^s(m)} \mathbb{E} \|\hat{\Phi}_\delta - f\|_p \leq cn^{-\frac{s}{2s+d}}, \quad (8)$$

(ii) If $1 \leq p < 2$, $x_0 \in \mathcal{M}$, $R > 0$, and $\delta = n^{-\frac{1}{2s+d}}$, then

$$\sup_{f \in B_{p\tau}^s(m, x_0, R)} \mathbb{E} \|\hat{\Phi}_\delta - f\|_p \leq cn^{-\frac{s}{2s+d}}, \quad (9)$$

(iii) If $\delta = \left(\frac{\log n}{n}\right)^{\frac{1}{2s+d}}$, then

$$\sup_{f \in B_{\infty\tau}^s(m)} \mathbb{E} \|\hat{\Phi}_\delta - f\|_\infty \leq c \left(\frac{\log n}{n}\right)^{\frac{s}{2s+d}}, \quad (10)$$

ADAPTATION

$$\delta = \left(\frac{\log n}{n} \right)^{\frac{1}{2s+d}}, \quad \delta = n^{-\frac{1}{2s+d}}$$

Suppose the knowledge of the regularity ? How to do without it ?

Different methods :

- Lepski's Method
- Penalization Bayesian methods
- Wavelet thresholding

ADAPTATION-LEPSKI'S METHOD

$$\hat{\delta}(x) := \sup\{\delta, |\hat{\Phi}_{\delta}(x) - \hat{\Phi}_{\delta'}(x)| \leq \sqrt{\frac{c \log n}{n[\delta']^d}}, \forall \delta' \leq \delta\}$$



ADAPTATION-BAYESIAN METHODS WITH HEAT KERNEL PRIOR CASTILLO-KERKYACHARIAN-P.

Prior : l- Gaussian process

$$K_t(x, y) = P_t(x, y) = \sum_k e^{-t\lambda_k} \left[\sum_l e_k^l(x) e_k^l(y) \right]$$
$$W^t(x) = \sum_k e^{-t\lambda_k} \left[\sum_l e_k^l(x) \xi_{kl} \right]$$

- $W^T(x) = \sum_k e^{-T\lambda_k/2} \sum_l \xi_{kl} e_k^l(x)$ with $T \sim t^{-a} e^{-t^{-d/2} \log^q(1/t)}$, $q = 1 + \frac{d}{2}$, ξ_{kl} i.i.d. $N(0, 1)$
- W^T seen as prior on $(\mathbb{L}_2, \|\cdot\|_2)$

Suppose $f_0 \in B_{2,\infty}^s(\mathcal{M})$ (Alfors condition). For R large enough, as $n \rightarrow +\infty$,

$$P_{f_0}^n(\Pi(\rho(f, f_0) \geq R \left(\frac{\log n}{n} \right)^{s/(2s+d)} \mid X^{(n)})) \rightarrow 0$$

ADAPTATION : WAVELET THRESHOLDING

We have some restriction on the space:

- 1 (M, ρ) is compact.
- 2 (M, ρ) is an Ahlfors regular space: $\exists 0 < a < b < \infty, \exists 0 < d < \infty,$

$$\forall x \in M, 0 < r < \text{diam}(M), \quad ar^d \leq \mu(B(x, r)) \leq br^d$$

From the wavelet construction we know

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in \mathcal{X}_j} \beta_{j,\xi} \psi_{j,\xi}; \quad \beta_{j,\xi} = \int_M f \tilde{\psi}_{j,\xi} = \mathbb{E}(\tilde{\psi}_{j,\xi}(X)); \quad X \sim f d\mu$$

Let $X_1, X_2, \dots, X_n$ i.i.d. such that $X \sim f(x)dx$, $0 \leq f \leq A$ ($A \geq 4$) (A is known) We suppose $f \in B_{r,\tau}^s$, $1 \leq r, \tau \leq \infty$, $0 < s$. The parameters s, r, τ are unknown but we assume $s' = s - \frac{d}{r} > 0$. Let us define $\hat{\beta}_{j,\xi}$ and J_n ,

$$\hat{\beta}_{j,\xi} = \frac{1}{n} \sum_{i=1}^n \tilde{\psi}_{j,\xi}(X_i), \quad b^{-J_n d} \sim \frac{\log n}{n}.$$

As well, let

$$\lambda_n = \kappa \sqrt{\frac{\log n}{n}} \quad \kappa = c\sqrt{8A}.$$

c is a universal constant and we define the estimator $\hat{f}_n$ of f .

$$\hat{f}_n = \sum_{0 \leq j \leq J_n} \sum_{\xi \in \mathcal{X}_j} \hat{\beta}_{j,\xi} 1_{|\hat{\beta}_{j,\xi}| > 2\lambda_n} \psi_{j,\xi}.$$

Then there exist constants $C(p, r, s)$, $C(r, s)$ only depending on the parameters in parenthesis such that:

1

$$\mathbb{E}(\|\hat{f}_n - f\|_\infty) \lesssim C(s, r) \|f\|_{B_{r,\tau}^s}^{\frac{d}{2(s-d(\frac{1}{r}-\frac{1}{2}))}} \left(\frac{\log n}{n}\right)^{\frac{s-\frac{d}{r}}{2(s-d(\frac{1}{r}-\frac{1}{2}))}}.$$

2 If $1 \leq p < \infty$, and $s \geq \frac{dp}{2} \left(\frac{1}{r} - \frac{1}{p}\right)$,

$$\mathbb{E}(\|\hat{f}_n - f\|_p) \lesssim C(p, r, s) \|f\|_{B_{r,\tau}^s}^{\frac{d}{2s+d}} \log n \left(\frac{\log(n)}{n}\right)^{\frac{s}{2s+d}}.$$

3 If $2 \leq p < \infty$, and $s < \frac{dp}{2} \left(\frac{1}{r} - \frac{1}{p}\right)$,

$$\mathbb{E}(\|\hat{f}_n - f\|_p) \lesssim C(p, r, s) \|f\|_{B_{r,\tau}^s}^{\frac{d(\frac{1}{2}-\frac{1}{p})}{s-d(\frac{1}{r}-\frac{1}{2})}} \log(n) \left(\frac{\log(n)}{n}\right)^{\frac{s-d(\frac{1}{r}-\frac{1}{p})}{2(s-d(\frac{1}{r}-\frac{1}{2}))}}.$$

RESEMBLANCES...



RESEMBLANCES...



-  P. Baldi, G. Kerkycharian, D. Marinucci, D. Picard, Adaptive density estimation for directional data using needlets, Ann. Statist. 37 (2009), no. 6A, 3362–3395.
-  I. Castillo, G. Kerkycharian, D. Picard, Thomas Bayes' walk on manifolds, Probab. Theory Relat. Fields, 158 (2014), no. 3-4, 665–710.
-  R. Coifman, G. Weiss, Analyse Harmonique Non-commutative sur Certains Espaces Homogenes. Lecture Notes in Math. Vol. 242. Springer, Berlin 1971.
-  T. Coulhon, G. Kerkycharian, P. Petrushev, Heat Kernel Generated Frames in the Setting of Dirichlet Spaces, J. Fourier Anal. Appl. 18 (2012), no. 5, 995–1066.
-  D. L. Donoho, I. M. Johnstone, G. Kerkycharian, D. Picard, Density estimation by wavelet thresholding, Ann. Statist. 24 (1996), no. 2, 508–539.

-  M. Frazier, B. Jawerth, G. Weiss, Littlewood-Paley theory and the study of function spaces, CBMS No 79 (1991), AMS.
-  W. Härdle, G. Kerkycharian, D. Picard, A. Tsybakov, Wavelets, approximation, and statistical applications, Lecture Notes in Statistics, 129, Springer-Verlag, New York, 1998.
-  G. Kerkycharian, P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces, Trans. Amer. Math. Soc. 367 (2015), 121–189.
-  G. Kerkycharian, P. Petrushev, D. Picard, T. Willer, Needlet algorithms for estimation in inverse problems, Electron. J. Stat. 1:30–76 (electronic), 2007.
-  G. Kerkycharian, P. Petrushev, Y. Xu, Gaussian bounds for the weighted heat kernels on the interval, ball and simplex, Constr. Approx. to appear. arXiv:1801.07325



G. Kerkycharian, P. Petrushev, Y. Xu, Gaussian bounds for the heat kernels on the ball and simplex: Classical approach, Studia Math. to appear. arXiv:1801.07326